Extended Pair Approximation of Evolutionary Game on Complex Networks

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We investigate how network structure influences evolutionary games on networks. We extend the pair approximation to study the effects of degree fluctuation and clustering of the network. We find that a larger fluctuation of the degree is equivalent to a larger mobility of the players. In addition, a larger clustering coefficient is equivalent to a smaller number of neighbors.

§1. Introduction

Evolutionary games on networks have recently attracted attention in evolutionary biology, behavioral science and statistical physics. In particular, an intensively studied question is how network structure influences the evolution of cooperative behavior in a dilemma situation (for regular lattices[1)–6)] and complex networks[7)–14)]). Nowak and May observed that cooperation behavior can be enhanced in the prisoner's dilemma game on a lattice network.[2)] Contrastingly, Hauert and Doebeli found that network structure often inhibits cooperation behavior in the snowdrift game.[6)] Because most of these papers are based on numerical simulations, it is not clear how the network architecture affects the evolution in general cases. The purpose of this study is to establish a theoretical formula describing the network effect in evolutionary game theory.

In this study, we focus on the influence of the average degree, degree fluctuations and clustering structure on the asymptotic result. To analyze these effects, we apply the pair approximation technique. The pair approximation is a useful tool for analysis of model ecosystems, because it can predict the population dynamics more accurately than the mean-field approximation.[15)–18)] The procedure for the pair approximation is often so complicated that it is carried out by numerical calculations. Here, to obtain an analytical solution, we propose novel procedures for the pair approximation.

§2. Games on Networks

A game on network is defined as follows. Let us consider a static network. Each node of network is occupied by an individual. Every individual plays games with its neighbors and reproduces depending on the score of games. Furthermore, we

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introduce migration of the players on the static network to clear network effects.

In this study, we consider four types of networks. First, we study the case of random regular graphs, in which all nodes have the same degree (the number of neighbors) z, and links are random (i.e. no correlations, clustering, etc.).[19] We assume $z > 2$ so that the networks are not divided into many disconnected components. Next, we expand the study to the case of networks in which the degree is distributed: Erdős-Rényi random graphs[19] and Barabási-Albert scale-free networks.[20] Finally, to study the clustering effect, we consider random regular graphs with a high level of clustering.[21]

Consider a symmetric game with two strategies, A and B, with the payoff matrix

$$
\begin{array}{cc}
 & \begin{array}{cc} A & B \end{array} \\
\begin{array}{c} A \\ B \end{array} & \left(\begin{array}{cc} a & b \\ c & d \end{array} \right)
\end{array} .
$$

Here we assume $0 < a, b, c, d < 1$. A player uses either strategy A or B. The fitness of players A and B with i neighbors with the same strategy (A and B, respectively) is given by

$$
f_A(i) = 1 - w + w \left(a\frac{i}{z} + b\frac{z-i}{z} \right), \ \ f_B(i) = 1 - w + w \left(c\frac{z-i}{z} + d\frac{i}{z} \right),
$$

where w is a parameter that measures the intensity of selection.[1]

Let us assume the following update rule for evolutionary dynamics. At each time step, with probability $1-q$, reproduction process occurs as follows. (i) An individual is selected at random with a probability proportional to its fitness. (ii) The selected individual is duplicated and it replaces a random neighbor. With probability q, a diffusion process occurs as follows. (i) Two neighboring players are selected at random. (ii) Their locations are exchanged. The parameter q, which is between 0 and 1, measures the intensity of the mobility of players. In the limit $q \to 1$, the dynamics becomes well mixed.

§3.   Random regular graphs

First, we present the theoretical results for random regular graphs obtained with the pair approximation. Let X and Y be the following conditional probabilities:

$$
X = p_{AA}/p_A, \ Y = p_{BB}/p_B. \tag{3·1}
$$

Here, p_* is the concentration of player $*$, and p_{**} represents the doublet density of two neighboring players. We have $p_A = p_{AA} + p_{AB}/2$ and $p_B = p_{BB} + p_{AB}/2$. In the pair approximation, the system can be described by the two variables X and Y alone:

$$
p_A = \frac{1-Y}{2-X-Y},
$$

$$
p_B = \frac{1-X}{2-X-Y},
$$

$$p_{AA} = \frac{X(1-Y)}{2-X-Y}, \tag{3·2}$$

$$p_{AB} = \frac{2(1-X)(1-Y)}{2-X-Y},$$

$$p_{BB} = \frac{Y(1-X)}{2-X-Y}.$$

The probability that a player A has i neighbors with strategy A is given by

$$p_A(i) = \left(\begin{array}{c} z \\ i \end{array} \right) X^i (1-X)^{z-i}.$$

The strategy A replaces the strategy B only if the player selected to reproduce is A and the replaced neighbor is B. The probability that this event occurs is given by

$$
\begin{aligned}
P_{B \to A} &= \frac{p_A}{\Phi} \sum_{i=0}^{z} \frac{z-i}{z} f_A(i) p_A(i) \\
&= \frac{p_{AB}}{2\Phi} \left\{ 1 - w + w[b + (1 - 1/z)(a-b)X] \right\},
\end{aligned}
\tag{3·3}
$$

where we have used Eqs. (3·1) and (3·2) to derive the final part. Here Φ is a normalization constant, which is given by the average fitness over all individuals:

$$\Phi = 1 - w + w\{p_A[b + (a-b)X] + p_B[c + (d-c)Y]\}. \tag{3·4}$$

In the same way, the probability that the strategy B replaces the strategy A is given by

$$P_{A \to B} = \frac{p_{AB}}{2\Phi} \left\{ 1 - w + w[c + (1 - 1/z)(d-c)Y] \right\}. \tag{3·5}$$

From Eqs. (3·3) and (3·5), we obtain

$$\dot{p}_A = \frac{p_{AB}}{2\Phi} w[b - c + (1 - 1/z)(a-b)X - (1 - 1/z)(d-c)Y]. \tag{3·6}$$

The necessary condition for equilibrium is $\dot{p}_A = 0$. This condition is simplified as

$$(a-b)X + (c-d)Y = \frac{c-b}{1-1/z} \tag{3·7}$$

From Eqs. (3·3), (3·4), (3·5) and (3·7), we obtain the following approximate relation:

$$P_{B \to A} = P_{A \to B} = p_{AB}/2 + O(w/z). \tag{3·8}$$

In addition, through the reproduction process, the rate of change of the doublet density is given by

$$
\begin{aligned}
P_{AB \to AA} &= [1 + (z-1)(1-Y)]P_{B \to A} \\
P_{AA \to AB} &= (z-1)X P_{A \to B}.
\end{aligned}
\tag{3·9}
$$

Here, we have used the fact that the replaced player has $z-1$ neighbors, other than the player selected to reproduce. Then, through the diffusion process, the rate of change of the doublet density is given by

$$
\begin{aligned}
P'_{AB \to AA} &= (z-1)(1-Y)p_{AB}, \\
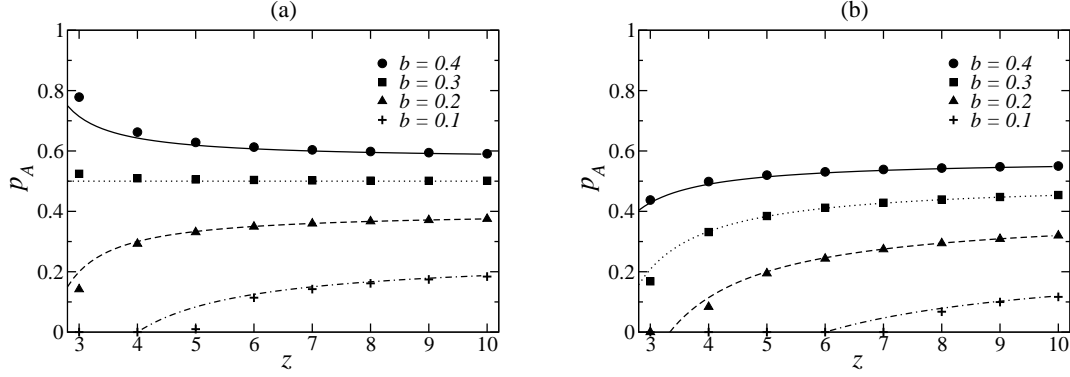P'_{AA \to AB} &= (z-1)X p_{AB}.
\end{aligned}
\tag{3·10}
$$

Fig. 1.   The density p_A of player A plotted as a function of the degree z for a random regular graph (a) $q = 0$, (b) $q = 0.2$. The game parameters are set as $b = 0.1$, 0.2, 0.3 and 0.4 from bottom to top for fixed $a = 0.7$, $c = 1$, $d = 0$ and $w = 0.5$. The total number of players is fixed to 10000. In all simulations, p_A is obtained by averaging over the last 10,000 time steps after the first 10,000 ones, and each data point results from 10 different network realizations. The curves represent the theoretical predictions (3·13).

Here, we have used the fact that each node of the doublet has $z - 1$ links excluding the link between the doublet. We have

$$\dot{p}_{AA} = (1 - q)P_{AB \to AA} + qP'_{AB \to AA} - (1 - q)P_{AA \to AB} - qP'_{AA \to AB}. \qquad (3\cdot11)$$

The equilibrium state must satisfy $\dot{p}_{AA} = 0$. Using Eqs. (3·8), (3·9) and (3·10), we obtain

$$X + Y = 1 + \frac{1 - q}{(1 + q)(z - 1)} \qquad (3\cdot12)$$

This result is an approximation valid for $qw/z \ll 1$. The equilibrium values of X and Y are obtained by solving eqs. (3·7) and (3·12). From Eq. (3·2), the equilibrium density of player A is given by

$$p_{A*} = \frac{b - d}{c - a + b - d} - \frac{c - a - b + d + q(c + a - b - d)}{(c - a + b - d)[(1 + q)z - 2]} \qquad (3\cdot13)$$

The stability is determined by calculating the Jacobi matrix of the dynamics of X and Y from (3·6) and (3·11). After using (3·8) and some algebra, we obtain that if $c - a + b - d > 0$ and $0 < p_{A*} < 1$ then p_{A*} is stable. The results are summarized in Table I.

For example, when $c > a$ and $b > d$ (the chicken game), the equilibrium p_{A*} is stable for large z (see the first and second rows in Table I). When $c < a$ and $b < d$ (the assurance game), the equilibrium p_{A*} is unstable and thus the density of players should approach to 0 or 1, depending on the initial conditions (see the third and fourth rows in Table I). Furthermore, when $b < d < a < c$ (the prisoner's dilemma game), either $p_{A*} < 0$ (see the fifth row in Table I) or $p_{A*} > 1$ (see the ninth row in Table I). Thus, all players should use strategy B (i.e. uncooperative behavior) ultimately and strategy A (i.e. cooperative behavior) is never sustainable. In the limit $z \to \infty$, Eq. (3·13) approaches $(b - d)/(c - a + b - d)$, which is the Nash

Table I. Equilibrium and its stability. The solid and dotted curves indicate stable and unstable equilibrium, respectively. Here, we have $z_0 = \frac{c-a+b-d+q(c+a-b-d)}{(b-d)(1+q)}$ and $z_1 = \frac{c-a+b-d-q(c+a-b-d)}{(c-a)(1+q)}$.

$c-a$	$b-d$	$c-a+b-d$	$c-a-b+d$ $+q(c+a-b-d)$	equilibrium
+	+	+	+	(phase diagram: p_A vs z, z_0)
+	+	+	−	(phase diagram: p_A vs z, z_1)
−	−	−	+	(phase diagram: p_A vs z, z_1)
−	−	−	−	(phase diagram: p_A vs z, z_0)
+	−	+	+	(phase diagram: p_A vs z)
+	−	+	−	(phase diagram: p_A vs z, z_1, z_0)
−	+	+	+	(phase diagram: p_A vs z, z_1, z_0)
−	+	+	−	(phase diagram: p_A vs z)
+	−	−	+	(phase diagram: p_A vs z)
+	−	−	−	(phase diagram: p_A vs z, z_1, z_0)
−	+	−	+	(phase diagram: p_A vs z, z_1, z_0)
−	+	−	−	(phase diagram: p_A vs z)

equilibrium in conventional game theory. In the limit $q \to 1$, Eq. (3·13) corresponds to the mean-field approximation. Note that the mean-field approximation does not

coincide with the Nash equilibrium, because the number of opponent players is restricted. Figure 1 shows that this approximation agrees very well with the numerical simulations.

§4. Degree fluctuation

Let us now consider the case in which the degree is not uniform but exhibits a distribution $\rho(k)$. Then, the average degree is written

$$z = \langle k \rangle = \sum_k k\rho(k). \tag{4·1}$$

In this case, we need to extend the pair approximation. Here, we do not take degree correlation into account. Furthermore, we assume that the density p_A and the conditional probabilities X and Y do not depend on the degree. Without this assumption, the relation (3·2) no longer hold, and thus we need more than two variables to describe the system. Although three variables (e.g. p_A, p_{AA} and p_{AB}) are often used to account for the dependence of p_A on the degree,[17), 18)] the outcome is too complicated to give clear insight. In addition, the approximation with three variables yields results that are quantitatively similar to the approximation with two variables, because the dependence of p_A on the degree is weak, as seen below. Accordingly, we adopt the approximation with two variables X and Y.

If a link is selected at random, the distribution of the degree of the nodes to which the particular link leads is not $\rho(k)$ but rather $k\rho(k)$.[22), 23)] Thus, the average degree of the player replaced in the reproduction process and the two players of the exchanged doublet in the diffusion process is given by

$$\frac{\sum_k k^2 \rho(k)}{\sum_k k\rho(k)} = \frac{\langle k^2 \rangle}{\langle k \rangle}. \tag{4·2}$$

In this case, we should use (4·2) instead of z in (3·9) and (3·10) to calculate the equilibrium. As a result, instead of (3·12), we have

$$X + Y = 1 + \frac{1-q}{(1+q)(\langle k^2 \rangle/\langle k \rangle - 1)}. \tag{4·3}$$

Equation (4·3) can be obtained by substituting the effective mobility

$$q' = q + \frac{1-q^2}{\kappa + q} \tag{4·4}$$

for q in (3·12), where $\kappa = (\langle k^2 \rangle + \langle k \rangle^2 - 2\langle k \rangle)/(\langle k^2 \rangle - \langle k \rangle^2)$. The parameter κ is larger than 1 for $z = \langle k \rangle > 2$, and thus it decreases with the variance of the degree distribution. It is obvious that the additional part, $(1-q^2)/(\kappa+q)$, in (4·4) is positive (because $0 < q < 1$) and a decreasing function of κ. Consequently, increasing the variance of the degree is equivalent to increasing the mobility, q.

For examples, we consider Erdős-Rényi random graphs and Barabási-Albert scale-free networks. For Erdős-Rényi random graphs, the degree follows a Poisson distribution, $\rho(k) = e^{-z}z^k/k!$, which leads to $\langle k^2 \rangle = z(z+1)$. Thus, we obtain
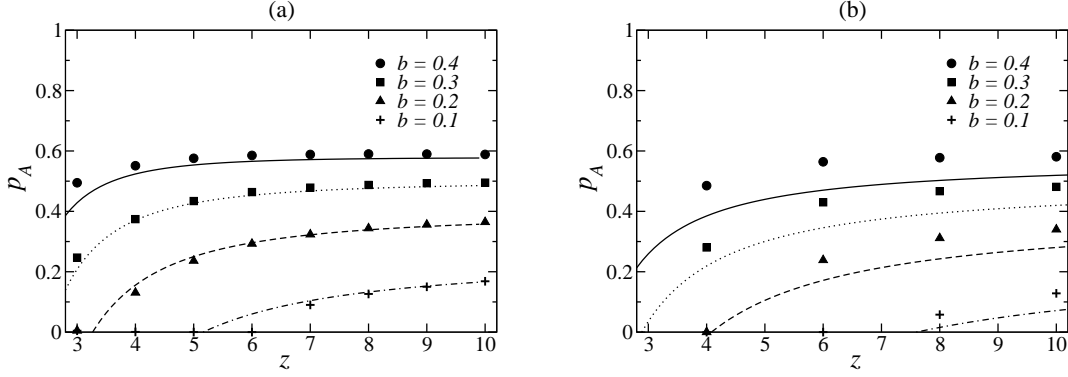
Fig. 2. The density of player A is plotted as a function of the average degree z for Erdős-Rényi random graphs (a) and for Barabási-Albert scale-free networks (b). The curves represent the theoretical prediction. Here $q = 0$, and the other parameters are the same as Fig. 1.
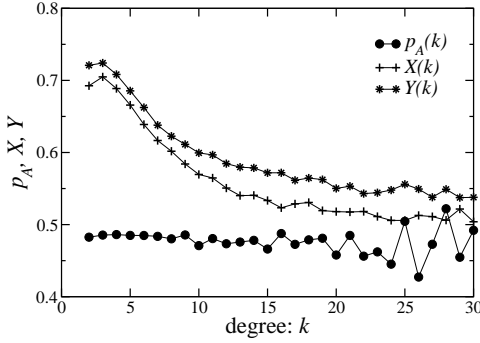

Fig. 3. The density p_A of player A and the conditional probabilities X and Y are plotted as functions of the degree k for Barabási-Albert scale-free networks. Here $a = 0.7$, $b = 0.4$, $c = 1$, $d = 0$, $w = 0.5$, $q = 0$ and $z = 4$. The curves are a guide for the eyes.
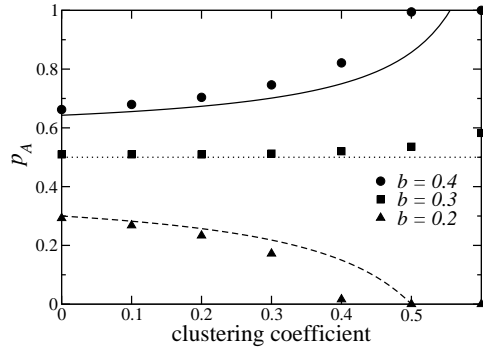

Fig. 4. The density of player A is plotted as a function of the clustering coefficient C for networks whose nodes have fixed degree $z = 4$. Here $q = 0$ and the other parameters are the same as Fig. 1.

$\kappa = 2z - 1$. For Barabási-Albert scale-free networks, the degree follows a power law distribution, $\rho(k) \propto k^{-\gamma}$, with the exponent $\gamma = 3$. In this case, $\langle k^2 \rangle \simeq z^2 \log N/4$ is obtained at leading order.[24] Thus, we obtain $\kappa = (z \log N + 4z - 8)/(z \log N - 4z)$. In the limit $N \to \infty$, we have $q' \to 1$, which means that the result approaches that of the mean-field approximation. For Erdős-Rényi random graphs, the theory agrees well with the numerical simulations [see Fig. 2(a)]. For Barabási-Albert scale-free networks, however, the agreement is not so good [see Fig. 2(b)]. This deviation mainly results from the fact that we ignored dependence of X and Y on the degree in this approximation. Figure 3 shows the numerical result, where p_A appears to be independent of the degree k, but X and Y decrease with k. Thus, a node with smaller degree tends to have homogeneous neighbors.

§5. Clustering effect

We now turn to a study of the clustering effect. For simplicity, we return to the case in which the degree is uniform. To this point, we have used network with a very small clustering coefficient. The clustering coefficient quantifies the probability that two vertices that are connected to the same node are also connected.[25] Thus, a network with a large clustering coefficient has many triangles, i.e., loop-like triplets. Ingnoring the triplet correlation without pair correlation biases, we assume that the density of three players on a triangle follows Kirkwood superposition approxima-tion:[26]–[28]

$$p_{AAB} : p_{ABB} \sim \frac{p_{AA}p_{AB}^2}{p_A^2 p_B} : \frac{p_{AB}^2 p_{BB}}{p_A p_B^2} = X : Y.$$

Note that in the original Kirkwood superposituon approximation, the normalization condition is violated.

Recall that the configuration of strategies changes only if two neighboring players have different strategies (i.e. A and B). The probability that these two neighboring players and another neighbor of one of them compose a triangle is given by C. In this case, the probabilities that the third player is A and B are p_{AAB} and p_{ABB}, respectively. Accordingly, instead of X and Y in Eqs. (3·7) and (3·12), we should use

$$X' = X(1-C) + C\frac{X}{X+Y}, \; Y' = Y(1-C) + C\frac{Y}{X+Y}. \qquad (5·1)$$

Here we should note that a pair of configurations p_{AAB} and p_{ABB} are normalized. In other studies, all configurations p_{AAA}, p_{AAB}, p_{ABB} and p_{BBB} have been normalized for similar approximations.[18),[28] Our approximation is better.

The formulation obtained using the replacement (5·1) can be also obtained by substituting z' and q' for z and q in (3·7) and (3·12) as follows:

$$
\begin{aligned}
q' &= q\left\{1 - \frac{(1-q^2)(z-1)}{(z-2)q+z}C + O(C^2)\right\}, \\
z' &= z\left\{1 - \frac{(1-q)(z-1)}{(z-2)q+z}C + O(C^2)\right\}.
\end{aligned}
$$

Thus, an increase of the clustering coefficient is equivalent to a decrease of z and q. In particular, when $q = 0$, this substitution is simplified exactly as

$$z' = z - C(z-1) \qquad (5·2)$$

In this case, the clustering effect is equivalent to the effect of decreasing the number z of neighbors. We present numerical results in Fig. 4. Here, to introduce the clustering structure into the random regular graphs, we used the edge exchange method.[21] In this method, two links are selected randomly, and they are rewired only when the new network configuration is connected and has a larger clustering coefficient. Figure 4 shows that our approximation agrees well with the numerical simulations. The deviation seen in the region of large C may be due to the fact that we ignored the effect from loops with more than three nodes.

§6. Summary

In conclusion, we have studied the network effect in general 2×2 game by using the pair approximation. First, for random regular graphs, our theoretical results are presented in (3·13) and Table I. Then, by the extended pair approximation, we developed a theory for networks with degree fluctuation and networks with large clustering. It was found that a fluctuation of the degree has the same effect as an increase in the mobility q, and a clustering structure has the same effect as a decrease in the number z of neighbors. Real social networks are more complex than the networks used in our numerical simulations, because they have degree correlation,[29] hierarchy and community structures.[30] We believe our method will be useful as a first step in analyzing such complicated situations. Furthermore, many other update rules for evolutionary dynamics are proposed.[3),6)] Some behavior seen in this paper depend on the update rule. The investigation of the dependence on the update rule is a future project.

Acknowledgements

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